PROPERTIES OF THE SOLUTIONS OF CERTAIN FUNCTIONAL DIFFERENTIAL EQUATIONS*

BY

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Introduction

The functional differential equations here under discussion have already been considered in a general way by Fr. Polussuchin who in her dissertation† reduced the solution of a more comprehensive class of equations to the solution of integral equations of the second kind. In the more detailed treatment of the present paper the existence of solutions of certain equations is established in the first part by the method of successive approximations. Some of the properties of the solutions whose existence is thus established are discussed in the second part.

I. An Existence Theorem

Consider the equation

$$(1) \qquad y^{(n)}(x) + \sum_{i=1}^{n} p_i(x) y^{(n-i)}(x) + r(x) y[\phi(x)] = 0,$$

where, within the closed interval (a, b) containing the point $x = c$, the functions $p_i(x)$ and $r(x)$ are continuous and $\phi(x)$ satisfies the conditions:

(1) $\phi(x)$ is continuous and within (a, b),

(2) $|\phi(x) - c| \leqq |x - c|$.

We proceed to establish the existence of solutions of (1) by the familiar method of successive approximations. We shall sketch the details for the case $n = 3$.

$$(2) \qquad y'''(x) + p_1(x) y''(x) + p_2(x) y'(x) + p_3(x) y(x) + r(x) y[\phi(x)] = 0.$$

We wish to establish the existence of a solution y of (2) such that $y(c) = u_0$, $y'(c) = v_0$, and $y''(c) = w_0$, where u_0, v_0, and w_0 are arbitrarily assigned numbers.

* Presented to the Society April 26 and December 31, 1919.

† *Über eine besondere Klasse von differentialen Funktionalgleichungen.* Inaugural-dissertation, Zürich, 1910.

Let $P_i (i = 1, 2, 3)$ and R be four numbers such that within (a, b)

$$|p_i(x)| \leqq P_i \qquad \text{and} \qquad |r(x)| \leqq R.$$

Put $u = y$, $v = y'$, and $w = y''$. Then (2) becomes

$$u'(x) = v(x); \qquad v'(x) = w(x);$$

$$w'(x) = -p_1(x)w(x) - p_2(x)v(x) - p_3(x)u(x) - r(x)u[\phi(x)].$$

We define the three series of functions $u_j(x)$, $v_j(x)$, and $w_j(x)$ $(j = 1, 2, \cdots)$ in the following way:

$$u_1'(x) = v_0; \qquad v_1'(x) = w_0;$$
$$w_1'(x) = -p_1(x)w_0 - p_2(x)v_0 - p_3(x)u_0 - r(x)u_0;$$

and, in general,

$$u_j'(x) = v_{j-1}(x); \qquad v_j'(x) = w_{j-1}(x);$$

$$w_j'(x) = -p_1(x)w_{j-1}(x) - p_2(x)v_{j-1}(x) - p_3(x)u_{j-1}(x) - r(x)u_{j-1}[\phi(x)];$$

$$u_j(c) = u_0, \qquad v_j(c) = v_0, \qquad w_j(c) = w_0.$$

If we choose a number A such that $|u_0|$, $|v_0|$, and $|w_0|$ do not exceed A, then $|u_1(x) - u_0| \leqq A|x - c|$, and, in general,

$$|u_j(x) - u_{j-1}(x)| \leqq M^{j-2} A \frac{|x - c|^j}{j!} \qquad (j > 1),$$

$$|v_j(x) - v_{j-1}(x)| \leqq M^{j-1} A \frac{|x - c|^j}{j!} \qquad (j \geqq 1),$$

$$|w_j(x) - w_{j-1}(x)| < M^j A \frac{|x - c|^j}{j!} \qquad (j \geqq 1),$$

where

$$\sum_{i=1}^{3} P_i + R < M, \qquad M \geqq 1,$$

and therefore

$$P_1 M + P_2 + P_3 + R < M^2$$

and

$$P_1 M^2 + P_2 M + P_3 + R < M^3.$$

It should be borne in mind that

$$|u_j[\phi(x)] - u_{j-1}[\phi(x)]| \leqq M^{j-2} A \frac{|\phi(x) - c|^j}{j!} \leqq M^{j-2} A \frac{|x - c|^j}{j!},$$

when $j > 1$ and that a similar inequality holds when $j = 1$.

The conclusion as to the existence and uniqueness of a solution satisfying the given initial conditions at the point $x = c$ follows as in the case of differential equations.* This conclusion can obviously be extended to the equation (1) of order n. The conclusion still holds if we remove from the function $\phi(x)$ the restriction (2) and assume instead either that

$$M(b-a) < 1, M \geqq 1, \text{ or } b - a < 1, M < 1.$$

If y and z are two solutions of (1) such that each of the differences $y^{(i)} - z^{(i)}$ $(i = 0, 1, \cdots, n-1)$ vanishes somewhere within an interval containing the point $x = c$, it can be shown as in the analogous case for differential equations† that there is a minimum possible length for this interval. The equation

$$y''(x) + \frac{5}{2}y(x) + \frac{3}{2}y(\pi - x) = 0$$

has the solutions $y = \cos x$ and $y = \cos 2x$, and these functions, as well as their first derivatives, are equal for $x = 0$. This shows the importance of the restriction that the interval we are considering shall contain the point $x = c$, which must here be the point $x = \pi/2$.

Moreover no non-identically vanishing solution of (1) and its first $n-1$ derivatives can vanish within a sufficiently small interval containing the point $x = c$. Hence if a solution vanishes n times within this interval, it vanishes identically here.

The following theorem is a direct consequence of the preceding discussion:

THEOREM I: *Within a properly restricted interval containing the point $x = c$ equation* (1) *has exactly n linearly independent solutions.*

The same method can of course also be used to establish the existence of solutions of non-homogeneous equations of the form

$$y^{(n)}(x) + \sum_{i=1}^{n} p_i(x)y^{(n-i)}(x) + r(x)y[\phi(x)] = g(x).$$

II. Properties of the Solutions

Consider first the equation

$$y^{(n)}(x) + p(x)y^{(n-1)}(x) + r(x)y(k-x) = 0, \tag{3}$$

where n is even and $r(x) > 0$ for the values of x under consideration. Here the c of the preceding discussion is $k/2$. It follows from Theorem I that within a properly chosen interval (a, b) with $k/2$ as center equation (3) has n linearly independent solutions. If we select $n-1$ arbitrary, distinct points

* See, for example, Goursat, *Cours d'analyse, 3d ed.*, vol. 2, p. 377.

† Fite, these Transactions, vol. 19 (1918), p. 351.

within such an interval, there is therefore a solution y that vanishes at these points.

Now y must change sign at each of these points, since otherwise y' would vanish at least $n-1$ times in (a, b), and the first $n-1$ derivatives of y, as well as y itself, would vanish in this interval. But it has been shown that this can happen only in case y vanishes identically. Moreover y will not vanish at any other point of the interval.

We proceed to show that y will not vanish at any point outside of (a, b). Let x_1 be the extreme left hand point in (a, b) at which y changes sign. If y vanished at any point to the left of a, let x_0 be the nearest such point to a, and suppose that y is negative between x_0 and x_1. Then as x increases from values a little greater than x_0 to b, y', beginning with a negative value, would change sign $n-1$ times; and $y^{(n-2)}$ would change from positive to negative and back to positive. Moreover the change from positive to negative would take place to the left of a, since otherwise $y^{(n-2)}$ would change sign twice within (a, b) and therefore $y^{(n-1)}$ would vanish in this interval together with y and its first $n-2$ derivatives. Then $y^{(n-1)}$ must be positive in (a, b), and negative a little to the left of a. It would therefore vanish at some point c in (x_0, a), while $y^{(n)}(c)$ would be positive and $y(k-c)$ would be negative. Hence y would change sign for some value of x greater than b. But for values of x a little less than b, y and its first $n-1$ derivatives would all be positive. Hence as x increases from b, $y^{(n-1)}$ would vanish before y does. But for the value of x at which this happens, $y(k-x) < 0$. Hence $y^{(n)}(x) > 0$. But $y^{(n-1)}(x)$ cannot pass from a positive value to zero when $y^{(n)}(x) > 0$. Hence y does not vanish to the left of a. If it vanished to the right of b, a similar argument shows that it would also vanish to the left of a. Hence

THEOREM II: *If equation* (3) *is of even order, no solution that vanishes at* $n-1$ *distinct points within a sufficiently small interval with* $x = k/2$ *as center can vanish at any other point of the interval within which the existence of solutions was established in Part I.*

In case $p(x)$ and $r(x)$ are continuous, and $r(x) > 0$, for all finite values of x, the argument shows furthermore that y increases without limit as x increases from b. As an example we cite the equation $y''(x) + y(-x) = 0$, which has the solution $y = e^x - e^{-x}$.

Suppose now that y_1 and y_2 are linearly independent solutions of (3) when $n = 2$. We can assume that $y_1(k/2) = y_2(k/2)$. Then the solution $y = y_1 - y_2$ changes sign at $k/2$, but at no other point within the interval of existence of the solutions. That is, two independent solutions that are equal for $x = k/2$ are not equal for any other value of x within this interval. It follows from this that if three consecutive changes of sign of y_1 on the same side of $k/2$ are at x_1, x_2, x_3 $(x_1 < x_2 < x_3)$, then y_2 does not change sign in

both of the intervals (x_1, x_2) and (x_2, x_3). Since the points at which a solution changes sign are not affected by multiplying the solution by a constant, the statement just made concerning the changes of sign of y_1 and y_2 applies to any two independent solutions. Moreover—and this result is analogous to Sturm's theorem of separation for the roots of solutions of linear homogeneous differential equations of order 2,—if x_1 and x_2 are two consecutive zeros of y_1 on opposite sides of $k/2$, every independent solution, as y_2, changes sign in (x_1, x_2). If it changes sign more than once in this interval, the points at which these changes occur are on the same side of $k/2$, since otherwise y_1 would change sign between the two of these points that are adjacent to $k/2$. If it changes sign once within the interval between any two consecutive changes of sign of y_1, it changes sign an even number of times in this interval, except when $k/2$ is in the interval, and then the number of changes of sign is odd.

We shall now consider the equation

$$y^{(n)}(x) + r(x)y(k - x) = 0, \tag{4}$$

where $r(x)$ is continuous for all finite values of x and greater than the positive number h for all sufficiently large positive and negative values of x. We shall find certain points of similarity with the corresponding type of differential equations and certain striking points of difference. It is necessary in the argument to distinguish between odd and even values of n.

(*a*) *n odd:* Suppose that a solution y does not change sign an unlimited number of times. We can then choose a value $x_1 (> k)$ such that y is positive or zero for all values of x greater than x_1,* and does not change sign for values of x less than $k - x_1$. If $y \geqq 0$ for these latter values of x, then $y^{(n)} \leqq 0$ for $x > x_1$, and $y^{(n-1)}$ would not increase as x increases from x_1. If it were negative for any such value of x, it would be negative for all greater values, and y would ultimately change sign. But this is contrary to our supposition. Hence $y^{(n-1)}$ would approach a limit which is equal to, or greater than, zero; and therefore $y^{(n)}$ would come indefinitely near to zero. This in turn would require that y should come indefinitely near to zero as x decreases from $k - x_1$. But as x decreases in this way none of the first $n - 1$ derivatives of y can change sign more than a limited number of times, since otherwise $y^{(n)}$ would change sign for a value of x less than $k - x_1$, and y would change sign for a value of x greater than x_1. Hence for a properly chosen x_2 each of the first n derivatives of y would have a fixed sign for all values of $x < x_2$. For these values of x, y' would be positive and would approach zero as x approaches $-\infty$. Hence y'' would also be positive, as would also all the derivatives up to, and including, $y^{(n)}$. But this is contrary to the supposition that $y \geqq 0$ for $x > x_1$.

* The argument here and farther on assumes that y is not identically zero for large values of x. It is not difficult to justify this assumption.

If, on the other hand, $y \leqq 0$ for $x < k - x_1$, we should have $y^{(n)} \geqq 0$ for $x > x_1$. If $y^{(n-1)}$ became positive for any of these latter values of x, it would remain positive for all greater values and y would increase indefinitely. Then $y^{(n)}$ would approach $-\infty$ as x decreases indefinitely, and y would therefore, under the same circumstances, become positive, since n is odd. But this is contrary to our supposition. If $y^{(n-1)}$ did not become positive, it would approach a limit and $y^{(n)}$ would come indefinitely near to zero as x approaches $+\infty$. This would require that y come indefinitely near to zero as x approaches $-\infty$, and, as before, for values of x less than an assignable x_2, each of the first n derivatives of y would approach zero as x approaches $-\infty$. This would require y to approach zero as x approaches $+\infty$. Now, by supposition, $y^{(n)}$ is positive or zero and $y^{(n-1)}$ is negative for sufficiently large positive values of x, and the successive derivatives from $y^{(n)}$ down to y' inclusive are of alternate signs, since otherwise y would either increase without limit, or change sign, as x increases. This would require y' to be positive—a result which is inconsistent with the assumption that y be positive and approach zero as x approaches $+\infty$. In this argument we have assumed $n > 1$. But it is obvious that the same conclusion holds for $n = 1$. Hence *y changes sign an unlimited number of times in case n is odd.*

(*b*) *n even:* The situation is essentially different when n is even, for we have seen that there are solutions which change sign exactly $n - 1$ times.

Suppose now that a solution y changed sign just an even number of times. We could assume that $y \geqq 0$ for $x > x_1$ and $x < k - x_1$, provided that x_1 is a properly chosen positive number. For $x > x_1$ the derivatives $y^{(n)}, y^{(n-1)}, \cdots, y'$ would be alternately negative and positive, since if two consecutive ones were negative, y would ultimately change sign, and if two consecutive ones were positive, y would increase without limit, and hence, for sufficiently large negative values of x, $y^{(n)}$ would be less than an assignable negative quantity and y would change sign contrary to our supposition. But if these derivatives were alternately negative and positive, y' would be positive, and we would have the same contradiction as before. This completes the proof of

Theorem III: *If $r(x)$ is continuous for all finite values of x and greater than the positive number h for all sufficiently large positive and negative values of x, every solution of the equation*

$$y^{(n)}(x) + r(x) y(k - x) = 0$$

*changes sign an infinite number of times in case n is odd; if n is even every solution changes sign either an odd number of times, or an infinite number of times.**

* In order to compare these results with the results for differential equations of corresponding form, see Kneser, Mathematische Annalen, vol. 42 (1893), pp. 434–5, and Fite, loc. cit., p. 349.

For example, the equation

$$y''(x) + y(\pi - x) = 0$$

has the solutions $y = \sin x$ and $y = e^{x} - e^{\pi - x}$. No solution except the former changes sign an infinite number of times.

We now remove the restriction that r be greater than h and require merely that it be positive and continuous for the values of x under consideration. For any solution y of (4) we have

$$(5)\quad y(x) = y\left(\frac{k}{2}\right) + y'\left(\frac{k}{2}\right)\left(x - \frac{k}{2}\right) + \cdots + y^{(n-1)}\left(\frac{k}{2}\right)\frac{\left(x - \frac{k}{2}\right)^{n-1}}{(n-1)!} - r(x_0)\,y(k - x_0)\frac{\left(x - \frac{k}{2}\right)^{n}}{n!},$$

where x_0 is between x and $k/2$. If c is the first root of the equation

$$(6)\quad y\left(\frac{k}{2}\right) + y'\left(\frac{k}{2}\right)\left(x - \frac{k}{2}\right) + \cdots + y^{(n-1)}\left(\frac{k}{2}\right)\frac{\left(x - \frac{k}{2}\right)^{n-1}}{(n-1)!} = 0$$

that is greater than $k/2$ and y has no root for $k/2 < x < c$, then $y(k - c_0) \leqq 0$, where $k/2 < c_0 < c$, if we assume that either $y(k/2) > 0$ or $y(k/2) = 0$ and that the first derivative of y that does not vanish for $k/2$ is positive here. If d is the first root of (6) less than $k/2$, and y has no root for $d < x < k/2$, then $y(k - d_0) \leqq 0$, where $d < d_0 < k/2$, in case n is even. If (6) has no real root, and n is odd, the first root of y greater than $k/2$ is nearer to $k/2$ than the first root less than $k/2$.

THEOREM IV: *If equation* (6) *has a real root distinct from $k/2$ when n is even, y changes sign at a point nearer to $k/2$ than any such root of* (6). *When n is odd and* (6) *has a root greater than $k/2$, y changes sign at a point nearer to $k/2$ than any such root of* (6). *If this equation has no real root, and n is odd, the first root of y greater than $k/2$ is nearer to $k/2$ than the first root less than $k/2$.*

We have now to consider the solutions of (4) under the conditions that $r(x)$ is continuous for all finite values of x and less than the negative number $-h$ for all sufficiently large positive and negative values of x. As before, it is necessary to distinguish between odd and even values of n.

(*a*) *n odd:* Suppose that for $n > 1$ we have a solution y that changes sign just an even number of times, or not at all; that is, that $y \geqq 0$ for $x > x_1$ and $x < k - x_1$. For $x > x_1$, $y^{(n)} \geqq 0$, and if $y^{(n-1)}$ were positive or zero for any such value of x, y would ultimately increase without limit as x approaches $+\infty$. Then as x approaches $-\infty$, $y^{(n)}$ would increase without limit and $y^{(n-1)}$ would decrease without limit. Tracing the effect of this on the pre-

ceding derivatives down to the first one and to y, we see that the latter would ultimately change sign as x decreases from $k - x_1$. But this is contrary to our assumption. We assume then that $y^{(n-1)}$ is negative for $x > x_1$. As x increases, it would therefore approach a limit and $y^{(n)}$ would come indefinitely near to zero. This would require that y come indefinitely near to zero as x decreases from $k - x_1$. Moreover for all sufficiently large negative values of x each of the first n derivatives of y would have a fixed sign, and $y^{(n)}$ would come indefinitely near to zero as x decreases from $k - x_1$. Hence y would come indefinitely near to zero as x increases from x_1. Now we have seen that $y^{(n-1)} < 0$ when $x > x_1$. If then $y^{(n-2)}$ were zero or negative for such a value of x, it would be negative for all greater values of x, and y would change sign for a value of $x > x_1$. We must assume then that $y^{(n-2)} > 0$ for $x > x_1$. A continuation of this argument shows that under the given conditions y' would be positive for these values of x. But this is inconsistent with the conclusion just reached that y must come indefinitely near to zero as x increases from x_1. If $n = 1$ it would be impossible for y to change sign just an even number of times. We conclude therefore that y cannot change sign just an even number of times when n is odd.

Suppose now that y changed sign just an odd number of times—that is, that $y \geqq 0$ for $x > x_1$ and $y \leqq 0$ for $x < k - x_1$. Then $y^{(n)} \leqq 0$ and $y^{(n-1)} > 0$ for $x > x_1$. If $n > 1$ and $y^{(n-2)}$ were positive or zero for some such value of x, y would increase without limit as x approaches $+ \infty$, and $y^{(n)}$ would increase without limit as x approaches $- \infty$. This would require that y decrease without limit under the same circumstances. Then $y^{(n)}$ would decrease without limit as x approaches $+ \infty$. But this is inconsistent with the fact that $y^{(n-1)}$ remains positive under the same circumstances. We must assume then that $y^{(n-2)} < 0$ for $x > x_1$. This would require that $y^{(n-1)}$ approach zero and that $y^{(n)}$ come indefinitely near to zero as x approaches $+ \infty$. Then y would come indefinitely near to zero as x approaches $- \infty$, and y' and $y^{(n)}$ would be negative for all sufficiently large negative values of x. But this is inconsistent with the fact that y is positive for $x > x_1$. If $n = 1$, y' would come indefinitely near to zero as x increases from x_1, and therefore y would come indefinitely near to zero as x decreases from $k - x_1$. But this is inconsistent with the fact that, for $x < k - x_1$, y is negative or zero and y' is positive or zero. Hence *when n is odd y' changes sign an infinite number of times.*

(*b*) *n even:* Suppose first that $y \geqq 0$ for $x > x_1$ and $y \leqq 0$ for $x < k - x_1$. Then, for $x > x_1$, $y^{(n)} \leqq 0$ and $y^{(n-1)} > 0$. If $n > 2$ and $y^{(n-2)}$ were positive or zero for such a value of x, y would increase without limit as x approaches $+ \infty$ and $y^{(n)}$ would increase without limit as x approaches $- \infty$. This would require that y be positive for sufficiently large negative values of x,

contrary to our supposition. If $y^{(n-2)}$ were negative for $x > x_1$, $y^{(n-1)}$ would approach zero, and $y^{(n)}$ would come indefinitely near to zero, as x approaches $+\infty$. Then y would approach zero as x approaches $-\infty$. But this is impossible, since, for $x < k - x_1$, $y^{(n)} \geqq 0$. The argument is substantially the same when $n = 2$, since we could assume $y' > 0$ for $x > x_1$, and y greater than an assignable positive number for these values of x.

There are equations that have solutions, y, such that $y \geqq 0$ for $x > x_1$ and $x < k - x_1$. For example, the equation $y''(x) - y(-x) = 0$ has the solutions $y = e^x + e^{-x}$, which does not vanish at all, $y = \sin x$, which changes sign an infinite number of times, and $y = e^x + e^{-x} - a \sin x$, which changes sign a given even number of times for a properly chosen value of a.

THEOREM V: *Every solution of equation* (4) *when* $r(x)$ *is continuous for all finite values of* x *and less than the negative number* $-h$ *for all sufficiently large positive and negative values of* x *changes sign an infinite number of times in case* n *is odd; in case* n *is even every solution changes sign an infinite number of times or an even number.**

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* In contrast with this result, compare Kneser, loc. cit., p. 413, and Bôcher, these Transactions, vol. 3 (1902), p. 199, theorem IV, for the properties of differential equations of corresponding form.